B-math 2nd year Final Exam
Subject : Analysis III

Time : 3.00 hours Max.Marks 60.

1. Let $\vec{r} = x\ \vec{i} + y\ \vec{j}$ denote the position vector of a point in the set $D := \{(x,y) : 1 < x^2 + y^2 < 25\}$ and $r := \sqrt{x^2+y^2}$. Let $\vec{F}(x,y) := \partial_y(\log\ r)\ \vec{i} - \partial_x(\log\ r)\ \vec{j}$. Let $\alpha(t)$ be the parametrisation of a piecewise smooth simple closed curve $C \subset D$. Find all possible values of $\int\limits_C \vec{F}\cdot d\alpha$. (9)

2. Let S be the solid torus in $\mathbb{R}^3$ generated by rotating the disc $D := \{(x,z) : (x-2)^2 + z^2 \leq 1\}$ about the z-axis.
a) Find a parametric representation of S.
b) Let $\vec{F} = P\ \vec{i} + Q\ \vec{j} + R\ \vec{k}$ be a vector field on S, where $P := \frac{-y}{x^2+y^2}, Q := \frac{x}{x^2+y^2}, R := z$ for $(x,y,z) \in S$. Show that curl $\vec{F} = 0$.
c) Show that $\vec{F} \neq \nabla\phi$ for any C^1 function ϕ in S (Hint : consider the line integral of F along a suitable curve lying in S). Explain why this does not contradict the result in b). (4+2+5)

3. Let $(X,Y) : \mathbb{R}^2 \to \mathbb{R}^2, X(u,v) := u+v, Y(u,v) := v-u^2, (u,v) \in \mathbb{R}^2$.
a) Compute the Jacobian of this transformation.
b) Let T be the triangle in $\mathbb{R}^2$ with vertices $(0,0),(2,0),(0,2)$. Let S be the image of T under the map (X,Y). Calculate the area of S.
c) Let $Q := \{(x,y) : |x| + |y| \leq 1\}$ and $f : [-1,1] \to \mathbb{R}$ be continuous and define $g(x,y) := f(x+y)\ (x,y) \in Q$. Show that g is Reimann integrable on Q and $\iint\limits_Q g(x,y)dx\ dy = \int\limits_{-1}^{1} f(u)\ du$. (2+4+5)

4. Let $a > 0, n \geq 2$ and $S_n(a) := \{(x_1, \cdots, x_n) : |x_i| + |x_n| \leq a, i = 1, \cdots, n-1\} \subset \mathbb{R}^n$. Calculate the volume of $S_n(a)$. Hint : what is the relationship between the volumes of $S_n(a)$ and $S_n(1)$? (8)

5. Let $f_n, g : (0,\infty) \to \mathbb{R}, n \geq 1$, where f_n and g are Reimann integrable on every closed and bounded interval $[a,b]$. Let $|f_n(x)| \leq g(x), x \in (0,\infty)$. Suppose that $\int_0^\infty g(x)\ dx := \lim\limits_{n\to\infty} \int\limits_{\frac{1}{n}}^{n} g(x)\ dx < \infty$. Let $f_n(x) \to f(x)$ for

every $x \in (0, \infty)$. State suitable assumptions on the sequence $\{f_n\}$ so that the following result is true and prove your result :

$$\lim_{n \to \infty} \int_0^\infty f_n(x)\ dx = \int_0^\infty f(x)\ dx \tag{8}$$

6. Let $0 < a < b$ and $\Phi := \{(x, y) : x = a \cos \mathrm{t}, y = ((1 - u)b + ua) \sin \mathrm{t}$, $0 \leq u \leq 1, 0 \leq t \leq 2\pi\}$ be a 2-surface in $\mathbb{R}^3$.
a) Describe the boundary of the *set* Φ.
b) Show that $\int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2(t)} dt = 2\pi$.
Hint : Consider the integral $\int\limits_C F \cdot d\alpha$ for a suitable vector field F in $\mathbb{R}^2 \setminus \{0\}$, where α is an appropriate parametrisation of the boundary C of the set Φ.
(4+6)

7. Let $f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) := (e^x \cos \mathrm{y}, e^x \sin \mathrm{y})$.
a) Show that f is not injective.
b) Show that there is an unbounded open connected set in $\mathbb{R}^2$ where f is one to one.
c) Let $a = (0, \frac{\pi}{3})$ and $b = f(a)$. Show that f has an inverse g in a neighborhood of b and compute $Dg(b)$. (2+2+4)